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ISOTROPIC RANDOM CURRENT

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1. Introduction

The theory of isotropic random vector fields was originated by H. P. Robertson [1] in his theory on isotropic turbulence. He defined the covariance bilinear form of random vector fields which corresponds to Khinchin's covariance function in the theory of stationary stochastic processes. Although in the latter theory the essential point was made clear in connection with the theory of Hilbert space and that of Fourier analysis, we have no corresponding theory on isotropic random vector fields.

Robertson obtained a condition necessary for a bilinear form to be the covariance bilinear form of an isotropic random vector field. Unfortunately his condition is not sufficient; in fact, he took into account only the invariant property of the covariance bilinear form but not its positive definite property. A necessary and sufficient condition was obtained by S. Itô [2]. Although his statement is complicated, he grasped the crucial point. His result corresponds to Khinchin's spectral representation of the covariance function of stationary stochastic processes.

The purpose of this paper is to establish a general theory on homogeneous or isotropic random vector fields, or more generally the homogeneous or isotropic random currents of de Rham [3]. In section 2 we shall give a summary of some known facts on vector analysis for later use. In section 3 we shall define random currents and random measures. The reason we treat random currents rather than random p-vector fields or p-form fields is that we have no restrictions in applying differential operators d and δ to random currents. These operators will elucidate the essential point. In section 4 we define homogeneous random currents and give spectral representations. Here we shall explain the relation between homogeneous random currents and random measures. In section 5 we shall show a decomposition of a homogeneous random current into its irrotational part, its solenoidal part and its invariant part. In the next section we shall give a spectral representation of the covariance functional of an isotropic random current. The result here contains S. Itô's formula as a special case. The spectral measure in this representation is decomposed into three parts which correspond to the above three parts in the decomposition of a homogeneous random current. This relation was not known to S. Itô. In [4] we have shown that the Schwartz derivative of the Wiener process is a stationary random distribution which is not itself a process. A similar fact will be seen in section 7 with respect to the gradient of P. Lévy's Brownian motion [5] with a multidimensional parameter.

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2. p-vector

We shall here summarize some known facts on vector analysis which will be used in this paper. Let $X = R^n$ be the Euclidean n-space and $\{e_i\}$ be an orthonormal regular basis, that is,

$$(e_i, e_j) = \delta_{ij}\,; \qquad i, j = 1, 2, \cdots, n\,, \tag{2.1}$$

and the vectors $e_i, \cdots, e_n$, in this order, give a positive orientation. Then any point x of X is expressed uniquely as

$$x = \sum x_i e_i\,. \tag{2.2}$$

The tangent space T_x and its dual space T_x^*, that is, the space of differentials at any point x of X, are both isomorphic to the space X itself by the following correspondence,

$$\frac{\partial}{\partial x_i} \leftrightarrow dx_i \leftrightarrow e_i\,, \qquad i = 1, 2, \cdots, n\,. \tag{2.3}$$

Therefore, we may identify both the space of p-vectors at x and that of p-forms with the space $X^{[p]}$ of p-vectors with complex coefficients in X. X may be considered as a real part of $X^{[1]}$.

Any p-vector $a_p \in X^{[p]}$ is expressed uniquely as

$$a_p = \sum_{[i]} a_{i_1 \cdots i_p} e_{i_1} \wedge \cdots \wedge e_{i_p}\,, \tag{2.4}$$

where the coefficients $a_{i_1 \cdots i_p}$ are complex numbers and $[i]$ means that the summation sign $\sum$ refers not to all systems of suffixes, but to those which satisfy $i_1 < i_2 < \cdots < i_p$.

The following notation will often be used in this paper:

$$\delta\begin{pmatrix} i_1 & \cdots & i_s \\ j_1 & \cdots & j_s \end{pmatrix} \tag{2.5}$$

is equal to 1 or -1 according to whether $\{j_\nu\}$ is an even or odd permutation of $\{i_\nu\}$, and is equal to 0 in all other cases. We shall state the definitions of exterior product $a_p \wedge \beta_q$, adjoint multivector a_p^*, inner product (a_p, β_p) and generalized inner product $a_p \vee \beta_q$, which are independent of the choice of the orthonormal regular basis,

$$a_p \wedge \beta_q = \sum_{[i]\,[j]\,[k]} a_{i_1 \cdots i_p} \beta_{j_1 \cdots j_q} \delta\begin{pmatrix} i_1 \cdots i_p & j_1 \cdots j_q \\ k_1 \quad \cdot \quad \cdot \quad \cdot & k_{p+q} \end{pmatrix} e_{k_1} \wedge \cdots \wedge e_{k_{p+q}}\,, \tag{2.6}$$

$$a_p^* = \sum_{[i]\,[j]} \bar{a}_{i_1 \cdots i_p} e_{j_1} \wedge \cdots \wedge e_{j_{n-p}} \delta\begin{pmatrix} 1 \quad 2 \quad \cdot \quad \cdot \quad \cdot & n \\ i_1 \cdots i_p \; j_1 \cdots & j_{n-p} \end{pmatrix}, \tag{2.7}$$

$$(a_p, \beta_p) = \sum_{[i]} a_{i_1 \cdots i_p} \bar{\beta}_{i_1 \cdots i_p} = \overline{(a \wedge \beta^*)\,^*}\,, \tag{2.8}$$

$$a_p \vee \beta_q = (-1)^{(q-p)\,(n-q)} (a_p \wedge \beta_q^*)\,^*\,, \qquad q \geqq p\,. \tag{2.9}$$

The product $\vee$ is dual to $\wedge$ in the following sense

$$(a_p \vee \beta_q, \gamma_{q-p}) = (\beta_q, a_p \wedge \gamma_{q-p})\,. \tag{2.10}$$

Now we shall consider a p-vector field or equivalently a p-form field $\alpha(x)$. Although $\alpha(x)$, in its proper sense, maps each point x of X respectively to a p-vector or p-form at x, it is also considered as a mapping from X to $X^{[p]}$ by the correspondence (2.3). Therefore, $\alpha(x)$ will be expressible as

$$\alpha(x) = \sum_{[i]} \alpha_{i_1} \cdots {}_{i_p}(x)\, e_{i_1} \wedge \cdots \wedge e_{i_p}, \tag{2.11}$$

where the coefficients $\alpha_{i_1} \cdots {}_{i_p}(x)$ are complex-valued functions of x. The following operations are common in the theory of p-forms.

Differential operators d and δ,

$$d\alpha_p(x) = \sum_{k,\,[i]} \frac{\partial \alpha_p}{\partial x_k} e_k \wedge e_{i_1} \wedge \cdots \wedge e_{i_p}, \tag{2.12}$$

$$\delta \alpha_p(x) = (-1)^{np+n+1} (d\alpha_p^*)^*. \tag{2.13}$$

Inner product,

$$\langle \alpha_p, \beta_p \rangle = \int_{R^n} (\alpha_p, \beta_p)\, dx_1 \cdots dx_n. \tag{2.14}$$

3. Random current and random measure

In this paper we shall treat only complex-valued random variables with mean 0 and finite variance. The totality of such random variables constitutes a Hilbert space which we shall denote by $\boldsymbol{H}$. The inner product of two elements in $\boldsymbol{H}$ is the covariance between them. We shall always refer to the strong topology in $\boldsymbol{H}$.

A random current is an $\boldsymbol{H}$-valued current of de Rham. Let $\mathfrak{D}_p$ be the set of all C_∞ p-vector fields with a compact carrier. $\mathfrak{D}_0$ is nothing but the Schwartz $\mathfrak{D}$ space. $\mathfrak{D}_p$ is isomorphic to the ${}_nC_p$th power of $\mathfrak{D}_0$, so that it is a linear topological space. A random current of the pth degree is a continuous linear mapping from $\mathfrak{D}_{n-p}$ into $\boldsymbol{H}$. A continuous random p-vector field $U_p(x)$ induces a random current of the pth degree as follows,

$$U_p(\phi_{n-p}) = \int_{R^n} (U_p \wedge \phi_{n-p})^*\, dx_1 \cdots dx_n. \tag{3.1}$$

A sequence of random currents $\{U_p^{(m)}\}$ is said to converge to U_p if

$$U_p^{(m)}(\phi_{n-p}) \to U_p(\phi_{n-p}), \qquad \phi_{n-p} \in \mathfrak{D}_{n-p}. \tag{3.2}$$

The operations on random currents are defined in the same way as in the case of the currents of de Rham,

$$U_p^*(\phi_p) = (-1)^{p(n-p)}\, \overline{U_p(\phi_p^*)}, \tag{3.3}$$

$$dU_p(\phi_{n-p-1}) = (-1)^{p+1} U_p(d\phi_{n-p-1}), \tag{3.4}$$

$$\delta U = (-1)^{np+n+1} (dU_p^*)^*, \tag{3.5}$$

$$\alpha_q(x) \wedge U_p(\phi_{n-p-q}) = (-1)^{pq} U_p(\alpha_q \wedge \phi_{n-p-q}), \tag{3.6}$$

$$\alpha_q(x) \vee U_p(\phi_{n-p+q}) = (-1)^{q(p-q)} U_p(\alpha_q \vee \phi_{n-p+q}). \tag{3.7}$$

For a random current U_p we shall define (U_p, α_p) as a random Schwartz distribution,

$$(U_p, \alpha_p)(\phi) = U_p(\phi \cdot \alpha_p^*). \tag{3.8}$$

A random Schwartz distribution $M(\phi)$ is called a random measure with respect to a measure m if we have, for any pair $\phi, \psi \in \mathfrak{D}$,

$$[M(\phi), M(\psi)]_H = \int_{R^n} \phi(x) \overline{\chi(x)}\, m(dx), \tag{3.9}$$

where $[M(\phi), M(\psi)]_H$ denotes the inner product in H. Putting $M(E) = M(\chi_E)$, where χ_E is the characteristic function of the set E, we get a random set function which is additive in E. Further, we have

$$[M(E), M(E')]_H = m(EE'), \qquad M(\phi) = \int_{R^n} \phi(x) M(dx). \tag{3.10}$$

Particularly, if m satisfies

$$\int_{R^n} \frac{m(dx)}{(1+|x|^2)^k} < +\infty, \tag{3.11}$$

that is, m is slowly increasing, then $M(\phi)$ is also said to be slowly increasing. Under this condition, any rapidly decreasing function ϕ in Schwartz sense [6] belongs to $L^2(R^n, m)$, so that $M(\phi)$ can be defined.

A random current $M_p(\phi_{n-p})$ is called a random measure (of the pth degree) if there exists a complex-valued locally finite measure $m(dx;\, a_p, b_p)$ for every pair (a_p, b_p) such that we have

$$[(M_p, a_p)(\phi), (M_p, b_p)(\psi)]_H = \int \phi(x) \overline{\psi(x)}\, m(dx;\, a_p, b_p). \tag{3.12}$$

If $m(dx;\, a_p a_p)$ is a slowly increasing measure for every a_p, then M_p is also said to be slowly increasing.

4. Homogeneous random current: spectral representation

A translation $\tau_h\colon x \to x + h$ induces a translation σ_h of p-vector fields in the following usual way,

$$(\sigma_h \cdot \phi_p)(x) = d\tau_h[\phi_p(x+h)]. \tag{4.1}$$

By the identification (2.3), we can easily see that $d\tau_h$ is just the identity mapping. Thus we have

$$(\sigma_h \cdot \phi_p)(x) = \phi_p(x+h). \tag{4.2}$$

Let U_p be a random current. Then we shall define $\sigma_h U_p$ by

$$(\sigma_h U_p)(\phi_{n-p}) = U_p(\sigma_h^{-1} \phi_{n-p}) = U_p(\sigma_{-h} \phi_{n-p}). \tag{4.3}$$

The functional $\rho(\phi_p, \psi_p) = [U_p(\phi_p^*), U_p(\psi_p^*)]_H$ is called the covariance functional of U_p. The function defined by

$$\rho(\phi, \psi;\, a_p, b_p) = [(U_p, a_p)(\phi), (U_p, b_p)(\psi)]_H \tag{4.4}$$

is called the *covariance bilinear form* of U_p and is a generalization of the form Robertson used in his theory of turbulence. If the covariance functional of $\sigma_h U_p$ is independent of h, then U_p is said to be *homogeneous*. If this condition is satisfied, the covariance bilinear form of $\sigma_h U_p$ is independent of h. The converse is also true.

As in the theory of stationary stochastic processes we have the following theorem of spectral representation.

THEOREM 4.1. *The covariance bilinear form of any homogeneous random current is written as*

$$\rho(\phi, \psi; a_p, b_p) = \int_{R^n} \mathcal{F}\phi(y)\overline{\mathcal{F}\psi(y)}\, m(dy; a_p, b_p) \tag{4.5}$$

where $m(\Lambda; a_p, b_p)$ is a positive definite bilinear form in (b_p, a_p) and $m(\Lambda; a_p, a_p)$ is a slowly increasing nonnegative measure. Conversely the ρ defined by (4.5) *is the covariance bilinear form of a homogeneous current.*

Remark. $\mathcal{F}\phi$ is the Fourier transform of ϕ, that is,

$$(\mathcal{F}\phi)(y) = \int_{R^n} e^{-i2\pi(x,y)}\phi(y)\, dy. \tag{4.6}$$

For a p-vector field $\phi_p(y)$, we define

$$(\mathcal{F}\phi_p)(y) = \sum_{[i]} \mathcal{F}\phi_{i_1 \cdots i_p}(y)\, e_{i_1} \wedge \cdots \wedge e_{i_p}. \tag{4.7}$$

THEOREM 4.2. *A homogeneous random current U_p is the Fourier transform of a slowly increasing random measure M_p, which is called the spectral measure of U_p, that is,*

$$U_p(\phi_{n-p}) = M_p = M_p(\mathcal{F}\phi_{n-p}). \tag{4.8}$$

We can prove this theorem easily by remarking that $(U_p, a_p)(\phi)$ is a stationary random distribution [4] with a multidimensional parameter.

5. Homogeneous random current: canonical decomposition

In this section we shall discuss a decomposition of a homogeneous random current U_p into an irrotational current, a solenoidal current and an invariant current. This decomposition will be called the canonical decomposition of U_p. We shall start from the spectral representation. By using this, we can show the existence of the limit

$$\mathcal{M} U_p(\phi_{n-p}) = \lim_{A\to\infty} \frac{1}{A^n}\int_{-A}^{A} \cdots \int_{-A}^{A} \sigma_h U_p(\phi_{n-p})\, dh_1 \cdots dh_n. \tag{5.1}$$

We say that U_p is invariant or unbiased according to whether $\mathcal{M}U_p = U_p$ or 0. An unbiased homogeneous random current is called irrotational or solenoidal according to whether $dU_p = 0$ or $\delta U_p = 0$. We set

$$M_p^0(\Lambda) = M_p(\Lambda \cap \{0\}), \qquad M_p^u(\Lambda) = M_p(\Lambda - \{0\}). \tag{5.2}$$

Then we have $\mathcal{M}U_p = \mathcal{F}M_p^0$.

THEOREM 5.1.

(a) *A homogeneous random current U is irrotational if and only if $M_p^0 = 0$ and $y \wedge M_p^u(dy) = 0$.*

(b) *U is solenoidal if and only if $M_p^0 = 0$ and $y \vee M_p^u(dy) = 0$.*

The essential point of the proof is as follows.

$$\begin{aligned} dU_p(\phi_{n-p-1}) &= (-1)^{p+1} U_p(d\phi_{n-p-1}) = (-1)^{p+1} M_p(\mathcal{F}d\phi_{n-p-1}) \\ &= (-1)^{p+1} M_p(i2\pi y \wedge \mathcal{F}\phi_{n-p-1}). \end{aligned} \tag{5.3}$$

Now we shall introduce two random measures M_p^i and M_p^s by

$$M_p^i(dy) = \frac{y \wedge [y \vee M_p^u(dy)]}{|y|^2}, \qquad M_p^s(dy) = \frac{y \vee [y \wedge M_p^u(dy)]}{|y|^2}. \tag{5.4}$$

THEOREM 5.2. *$U_p^i = \mathcal{Q}M_p^i$, $U_p^s = \mathcal{Q}M_p^s$ and $U_p^0 = \mathcal{Q}M_p^0 = \mathcal{M}U_p$ are respectively irrotational, solenoidal and invariant, and we have*

$$U_p = U_p^i + U_p^s + U_p^0 . \tag{5.5}$$

By using the following identity

$$a_p = a_p' + a_p'' = \frac{y \wedge (y \vee a_p)}{|y|^2} + \frac{y \vee (y \wedge a_p)}{|y|^2} ; \qquad a_p', a_p'' = 0 ; \tag{5.6}$$

we can prove the theorem.

Now we shall define $W_p(\phi_{n-p})$ by the following procedure,

$$G(x, y) = \begin{cases} \dfrac{e^{-2i\pi(x, y)} - 1 + 2i\pi(x, y)}{|y|^2} & \text{if } |y| < 1 , \\ \dfrac{e^{-2i\pi(x, y)}}{|y|^2} & \text{if } |y| \geqq 1 ; \end{cases} \tag{5.7}$$

$$G(\phi_p, y) = \sum_{[i]} [\textstyle\int G(x, y)\, \phi_{i_1 \cdots i_p}(x)\, dx]\; e_{i_1} \wedge \cdots \wedge e_{i_p} ; \tag{5.8}$$

$$W_p(\phi_{n-p}) = \frac{1}{4\pi^2} M_p^u [G(\phi_{n-2}, y)] . \tag{5.9}$$

THEOREM 5.3.

$$d\,\delta W_p = U_p^i , \qquad \delta d W_p = U_p^s . \tag{5.10}$$

Therefore the canonical decomposition is written as

$$U_p = d\,\delta W_p + \delta d W_p + U_p^0 . \tag{5.11}$$

6. Isotropic random current

To begin with, we shall introduce some preliminary notation. Let G be the whole group of orthogonal transformations (with determinant ± 1) in $X = R^n$. We can carry out the same procedure for $g \in G$ as we did for the translation τ_h in section 4. Corresponding to (4.1) we have

$$(\sigma_g \phi_p)(x) = d\tau_g [\phi_p(g \cdot x)] . \tag{6.1}$$

Although $d\tau_g$ here is not the identity mapping, we can easily see that $d\tau_g = g^{-1}$. Therefore, we have

$$(\sigma_g \cdot \phi_p)(x) = g^{-1} \cdot \phi_p(g \cdot x) , \tag{6.2}$$

where $g^{-1}\phi_p$ is defined by

$$g^{-1}\phi_p = \sum_{[i]} \phi_{i_1 \cdots i_p} g^{-1} e_{i_1} \wedge \cdots \wedge g^{-1} e_{i_p} . \tag{6.3}$$

Generalizing this transformation σ_g, we define σ_g on a current u_p,

$$(\sigma_g u_p)(\phi_{n-p}) = u_p(\sigma_{g^{-1}} \phi_{n-p}) = u_p[g\phi_{n-p}(g^{-1}x)] . \tag{6.4}$$

σ_g is clearly commutative with the differential operators d and δ.

Now we shall define an isotropic random current. A homogeneous random current is said to be isotropic, if the covariance functional of $\sigma_g U_p$ is independent of $g \in G$. Even

if we replace the covariance functional by the covariance bilinear form in the above statement, we shall obtain an equivalent definition.

Let U_p be an isotropic random current. Since U_p is homogeneous, the covariance bilinear form is written as

$$\rho(\phi, \psi; a_p, b_p) = \int \mathcal{F}\phi\overline{\mathcal{F}\psi}\, m(dx; a_p, b_p) \tag{6.5}$$

by theorem 4.1. Since U_p is isotropic, we obtain

$$m(g\cdot dx; g\cdot a_p, g\cdot b_p) = m(dx; a_p, b_p)\,. \tag{6.6}$$

By making use of this property we obtain the following theorem.

THEOREM 6.1. *In an isotropic turbulence the* $m(dx; a_p, b_p)$ *in* (6.6) *is expressible as*

$$\begin{aligned} m(dx; a_p, b_p) = {} & (\theta\vee b_p, \theta\vee a_p)\, d\theta\, F_i(dr) \\ & + (\theta\wedge b_p, \theta\wedge a_p)\, d\theta\, F_s(dr) + (b_p, a_p)\, F_0(dx)\,, \end{aligned} \tag{6.7}$$

where $r = |x|, \theta = x/r, x \neq 0, d\theta$ *is the surface element of the unit sphere,* F_1 *and* F_2 *are both slowly increasing nonnegative measures on* $(0, \infty)$, *and* F_0 *is a nonnegative measure such that* $F_0(\Lambda) = 0$ *for* $0 \notin \Lambda$.

Conversely, $\rho(\phi, \psi; a_p, b_p)$, *if determined by* $m(dx; a_p, b_p)$ *of the form* (6.6), *is the covariance bilinear form of a certain isotropic random current.*

Remark. According to whether $p = 0$ or $p = n$, the first or the second term in (6.7) will disappear.

We shall sketch the proof. First we consider the case in which $m(dx; a_p, b_p)$ has a continuous density $f(x; a_p, b_p)$. Here $f(x; a_p, b_p)$ is a positive definite bilinear form in (b_p, a_p) and is invariant in the sense $f(gx; ga_p, gb_p) = f(x; a_p, b_p)$. If we introduce $F(x; \xi_1, \cdots, \xi_p, \eta_1, \cdots, \eta_p) = f(x; \xi_1\wedge\cdots\wedge\xi_p, \eta_1\wedge\cdots\wedge\eta_p)$, for real 1-vectors ξ_i, η_j, we obtain a function which is linear in ξ_i and in η_j and skew symmetric in each of $\{\xi_i\}$ and $\{\eta_j\}$. Since F is invariant under orthogonal transformations, it is a function of (ξ_i, ξ_j), (ξ_i, η_j), (η_i, η_j), (x, ξ_i), (x, η_j) and (x, x). Using these properties we can prove that f can be written in a form similar to (6.7).

We can discuss the case of general measure by approximating it by measures with continuous density in Helly's sense.

The following theorem shows that the decomposition in (6.6) corresponds to the canonical decomposition (5.5).

THEOREM 6.2. *In the case of an isotropic random current* U_p, U_p^i, U_p^s *and* U_p^0 *are all isotropic and orthogonal to each other in* $\boldsymbol{H}$. *The m-measures corresponding to these three parts are respectively the three parts in the decomposition* (6.7).

7. The case of $p = 1$

In the case of $p = 1$, the decomposition (6.7) becomes simpler,

$$m(dx; \alpha, \beta) = \sum_{\mu,\nu} \bar{\alpha}_\mu\beta_\nu\,[\theta_\mu\theta_\nu d\theta\, F_i(dr) + (\delta_{\mu\nu} - \theta_\mu\theta_\nu)\, F_s(dr) + F_0(dx)]\,, \tag{7.1}$$

where $\alpha = \sum \alpha_\mu e_\mu$ and $\beta = \sum \beta_\mu e_\mu$.

In the case of isotropic 1-vector fields the measures F_i and F_s will be bounded.

(a) *Robertson's isotropic turbulence.* This is an isotropic random 1-vector field in 3-

space. Our decomposition (7.1) is essentially the same as S. Itô's formula (2) but is somewhat simpler.

(b) *The gradient of P. Lévy's Brownian motion with a multidimensional parameter.* Let $B(x)$ be Lévy's Brownian motion with an n-dimensional parameter [5]. Then we have

$$[B(x), B(y)]_H = \frac{1}{2}(|x| + |y| - |x - y|). \tag{7.2}$$

Therefore, by simple computation we can obtain the following expression of the covariance bilinear form of the gradient dB of B,

$$\rho(\phi, \psi, \alpha, \beta) = \int_{R^n} \mathcal{F}\phi(x) \overline{\mathcal{F}\psi(x)}\, m(dx; \alpha, \beta), \tag{7.3}$$

where

$$m(dx; \alpha, \beta) = C_n \sum_{\mu, \nu} \alpha_\mu \beta_\nu \theta_\mu \theta_\nu d\theta\, r^{-(n-1)} dr, \tag{7.4}$$

where $r = |x|$, $\theta = x/r$, and C_n is a positive constant. This is the special case of (7.1) in which

$$F_i(dr) = C_n \cdot r^{-(n-1)} dr, \qquad F_s = F_0 = 0. \tag{7.5}$$

Since $C_n \cdot r^{-(n-1)} dr$ is an unbounded measure, dB is a random proper current, that is, a random current that is not itself a random 1-vector field. Since $F_s = F_0 = 0$ we see that dB is irrotational.

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